Chapter 3

ASYMPTOTES, SINGULAR POINTS AND CURVE TRACING

In this chapter we shall study:

- Plotting of Rational and Irrational functions.
- Intersection of Curve and Straight line at Infinity.

3.1 ASYMPTOTES

A straight line, at a finite distance from origin, is said to be an asymptote of the curve $y = f(x)$, if the perpendicular distance of the point P on the curve from the line tends to zero when x or y both tends to infinity.

OR

A straight line A is called an asymptote to a curve, if the distance δ from the variable point M of the curve to this straight line approaches zero as the point M tends to infinity. Shown as:

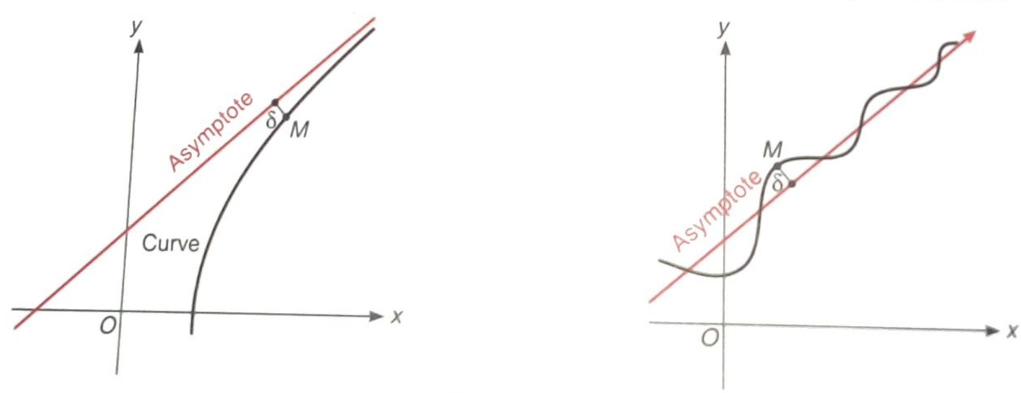


Fig. 3.1

Mathematically

Let $y = f(x)$ be a curve and let (x, y) be a point on it.

Tanget at (x, y) is given by;

$$Y - y = \frac{dy}{dx}(X - x)$$

$$Y = \frac{dy}{dx}\cdot X + \left(Y - x\frac{dy}{dx}\right) \qquad \text{...(i)}$$

Now, if asymptote exists, then $x \to \infty$

$$\Rightarrow \quad \frac{dy}{dx} \text{ and } \left(y - x\frac{dy}{dx}\right) \longrightarrow \text{finite limit say } m \text{ and } c$$

Say $$\frac{dy}{dx} \to m \quad \text{and} \quad y - \frac{x\,dy}{dx} \to c$$

$\therefore$ Eq. (i) reduces to, $\boldsymbol{Y = mX + c}$ is asymptote of equation.

Now we shall discuss the following cases

(i) Asymptote parallel to x-axis.
(ii) Asymptote parallel to y-axis.
(iii) Asymptote of algebraic curves or oblique asymptotes.
(iv) Asymptote by inspection.
(v) Intersection of curve and its Asymptotes.
(vi) Asymptote by Expansion.
(vii) The position of the curve with respect to asymptote.

3.1 (i) Asymptote parallel to x-axis

Let the equation of curve be,

$$(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + \ldots + b_ny^{n-1})$$
$$+ (c_2x^{n-2} + c_3x^{n-2}y + \ldots + c_ny^{n-2}) + \ldots + \ldots = 0 \qquad \ldots\text{(i)}$$

then it can be arranged in descending powers of x as follows:

$$a_0x^n + (a_1y + b_1)x^{n-1} + (a_2y^2 + b_2y + c_2)x^{n-2} + \ldots = 0 \qquad \ldots\text{(ii)}$$

Now, if $a_0 = 0$, *i.e.*, the term consisting x^n is absent, then $a_1y + b_1 = 0$, *i.e.*, coefficient of $x^{n-1} = 0$ will make two roots of Eq. (i) infinite as coefficients of both x^n and x^{n-1} are zero.

Hence, $\boldsymbol{a_1y + b_1 = 0}$ **is an asymptote parallel to x-axis.**

Again if; both x^n and x^{n-1} are absent, then $a_2y^2 + b_2y + c_2 = 0$, *i.e.*, coefficient of x^{n-2} being zero will make three roots of Eq. (ii) infinite hence, $\boldsymbol{a_2y^2 + b_2y + c_2 = 0}$ **will give two asymptote parallel to x-axis.**

Method to find asymptote parallel to x-axis

To find the asymptote parallel to x-axis equate the coefficient of highest power of x to zero.

If the coefficient is constant, then there is no asymptote parallel to x-axis (horizontal).

3.1 (ii) Asymptote parallel to y-axis

From above article, if we need an asymptote parallel to y-axis, equate the coefficient of highest power of y to zero.

If this coefficient is constant, then there is no asymptote parallel to y-axis (vertical).

EXAMPLE 1 *Sketch the curve* $y = \dfrac{1}{x-5}$

SOLUTION Here; $y(x-5) = 1$

$\therefore$ **Asymptote parallel to x-axis.**

$\Rightarrow$ $y = 0$ (equating highest power of $x = 0$)

Asymptote parallel to y-axis.

$\Rightarrow$ $x = 5$ (equating highest power of $y = 0$)

Thus, $x = 5$ and y-axis are asymptotes shown as in figure.

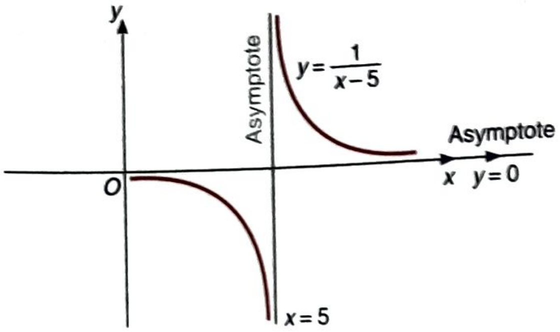


Fig. 3.2

EXAMPLE 2 *Show the curve $y = \tan x$ has an infinite number of vertical asymptote.*

SOLUTION $y = \tan x$

here $$y \to \pm\infty \quad \text{as} \quad x \to \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$$

or $$\tan x \to \infty \quad \text{as} \quad x \to \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$$

i.e., equating highest power of $y = 0$.

(as $y = \tan x \Rightarrow y \cot x = 1$, where $\cot x \to 0$).

Shown as:

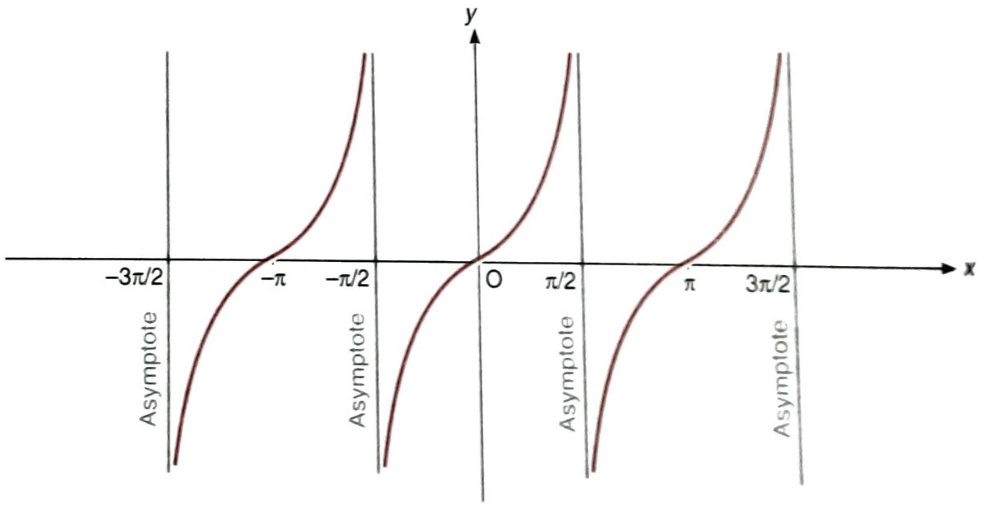


Fig. 3.3

EXAMPLE 3 *Show the curve $y = e^{1/x}$ has a vertical and horizontal asymptote.*

SOLUTION Here $$y = e^{1/x}$$

$\Rightarrow$ $$y \cdot e^{-1/x} = 0$$

or $$e^{-1/x} \to 0 \quad \text{as} \quad x \to 0 \qquad \left(\text{Since, } \lim_{x\to 0} e^{-1/x} \to 0\right)$$

From adjoining figure

$$y = e^{1/x}$$

$$\Rightarrow \quad \frac{1}{x} = \log y$$

$$\Rightarrow \quad x = \frac{1}{\log y}$$

which shows $x(\log y) = 1$ has an asymptote parallel to x-axis as

$$\log y = 0 \;\Rightarrow\; y = 1.$$

Thus, $y = e^{1/x}$ has two asymptote

$$x = 0 \quad \text{and} \quad y = 1.$$

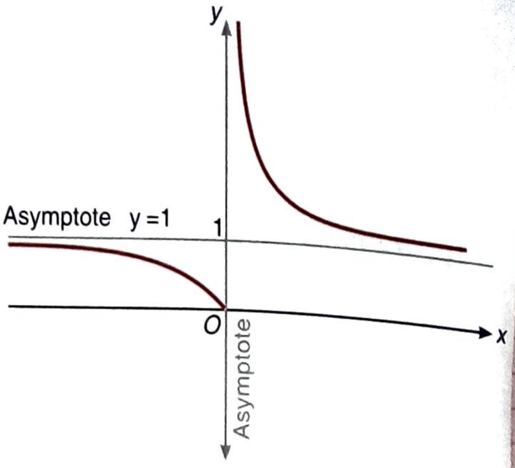


Fig. 3.4

3.1 (iii) Asymptote of algebraic curves or oblique asymptote

An asymptote which is not parallel to y-axis is called an oblique asymptote. Let $y = mx + c$ be an asymptote of $y = f(x)$, then

$$m = \lim_{\substack{x\to\infty \\ \text{or } x\to-\infty}} \frac{y}{x} \quad \text{and} \quad c = \lim_{\substack{x\to\infty \\ \text{or } x\to-\infty}} (y - mx)$$

Method to find oblique asymptote

Suppose $y = mx + c$ is an asymptote of the curve. Put $y = mx + c$ in the equation of the curve and arrange it in descending powers of x. Equate to zero the coefficients of two highest degree terms. Solve these two equations, find m and c. Put them in $y = mx + c$ to get asymptotes.

Note 1. Here, we will find non-parallel or non repeated asymptote only.
2. Neglect all imaginary values of m.

EXAMPLE 1 *Find the asymptotes to the curve* $y = x + \frac{1}{x}$ *and then sketch.*

SOLUTION Here, the given curve $y = x + \frac{1}{x}$

$$\Rightarrow \quad xy = x^2 + 1$$

$$\text{or} \quad x^2 - xy + 1 = 0$$

(i) Asymptote parallel to x-axis

Equating highest power coefficient of x to zero in $x^2 - xy + 1 = 0$

$$\Rightarrow \quad 1 = 0 \qquad \text{(which is not true)}$$

$\therefore$ no asymptote parallel to x-axis.

(ii) Asymptote parallel to y-axis

Equating highest power coefficient of y to zero in

$$x^2 - xy + 1 = 0$$

$\Rightarrow$ $$-x = 0$$

or $$x = 0 \qquad (i.e., \ y\text{-axis}) \text{ is asymptote for } y = x + \frac{1}{x}$$

(iii) Oblique asymptote

Let $$y = mx + c \quad \text{in} \quad x^2 - xy + 1 = 0$$

i.e., $$x^2 - mx^2 - xc + 1 = 0$$

$\Rightarrow$ $$x^2(1-m) - (c)x + 1 = 0$$

Equating highest and second highest power of x to zero

$$1 - m = 0 \quad \text{and} \quad c = 0$$

i.e., $$m = 1 \quad \text{and} \quad c = 0$$

$\therefore$ $$y = x$$

or is oblique asymptote to $y = x + \dfrac{1}{x}$.

Now to trace the curve;

(iv) Symmetric about origin (as odd function)

(v) Domain $\in R - \{0\}$.

(vi) Range $\in (-\infty, -2] \cup [2, \infty)$

(vii) $\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2}$ {using number line rule, + -1 − 1 + }

$$\frac{dy}{dx} > 0, \quad \text{when} \quad x < -1 \quad \text{or} \quad x > 1$$

$$\frac{dy}{dx} < 0, \quad \text{when} \quad -1 < x < 1 - \{0\}$$

which shows
$$\boldsymbol{y_{max}} \text{ at } \boldsymbol{x = -1}$$
$$\boldsymbol{y_{min}} \text{ at } \boldsymbol{x = 1}$$

(viii) Also, $$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} > 0, \quad \text{when} \quad x > 0 \qquad \text{(concave up)}$$

$$\frac{d^2y}{dx^2} < 0, \quad \text{when} \quad x < 0 \qquad \text{(concave down)}$$

Using above information we can trace $y = x + \frac{1}{x}$ as;

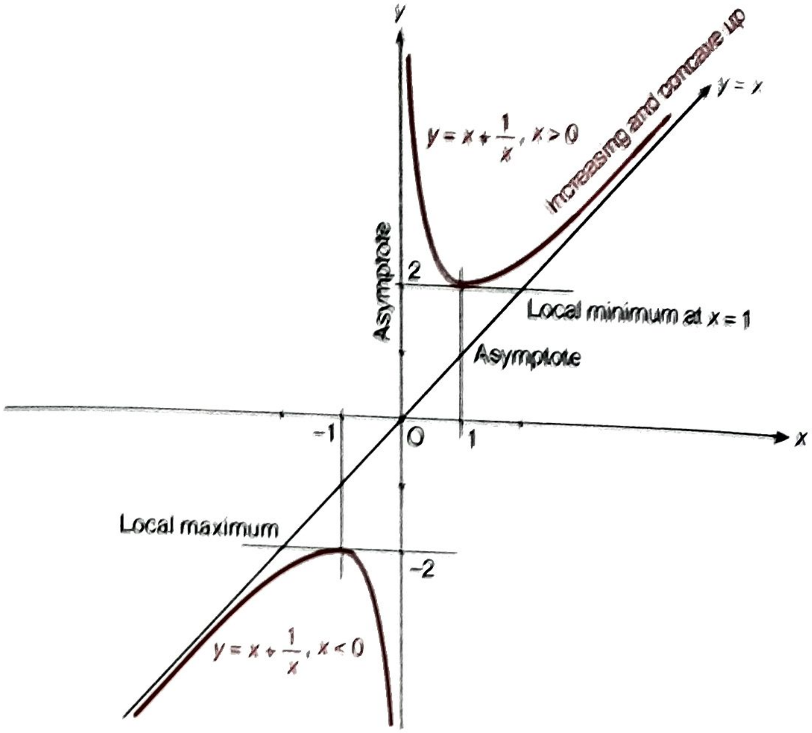


Fig. 3.5

EXAMPLE 2 *Find the asymptotes of the curve* $y = \dfrac{x^2 + 2x - 1}{x}$ *and hence, sketch.*

SOLUTION Here, the curve $y = \dfrac{x^2 + 2x - 1}{x}$ could be written as;

$$x^2 + 2x - yx - 1 = 0 \qquad \text{...(i)}$$

(i) No asymptote parallel to x-axis.

(ii) Asymptote parallel to y-axis. $\Rightarrow$ $x = 0$.

(iii) Oblique asymptote

Let $y = mx + c$ be oblique asymptote

$$\therefore \qquad x^2 + 2x - x(mx + c) - 1 = 0$$

$$x^2 - mx^2 + 2x - cx - 1 = 0$$

$$\Rightarrow \qquad x^2(1 - m) + x(2 - c) - 1 = 0$$

For oblique asymptote equate highest power and second highest power of x to zero.

i.e., Coefficient of $x^2 = 0 \Rightarrow m = 1$

Coefficient of $x = 0 \Rightarrow c = 2$

$\therefore$ $y = x + 2$ is oblique asymptote to $y = x - \frac{1}{x} + 2$

(iv) Neither symmetric about axis nor about origin.

(v) Domain $\in R - \{0\}$.

(vi) Range $\in R$.

(vii)
$$\frac{dy}{dx} = 1 + \frac{1}{x^2}$$

$$\frac{dy}{dx} > 0, \quad \text{for all} \quad x \in R - \{0\}.$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{2}{x^3}$$

(viii)
$$\frac{d^2y}{dx^2} > 0, \quad \text{when} \quad x < 0 \qquad \textbf{(concave down)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} < 0, \quad \text{when} \quad x > 0 \qquad \textbf{(concave up)}$$

Using above information, we can plot the curve $y = x - \dfrac{1}{x} + 2$ as;

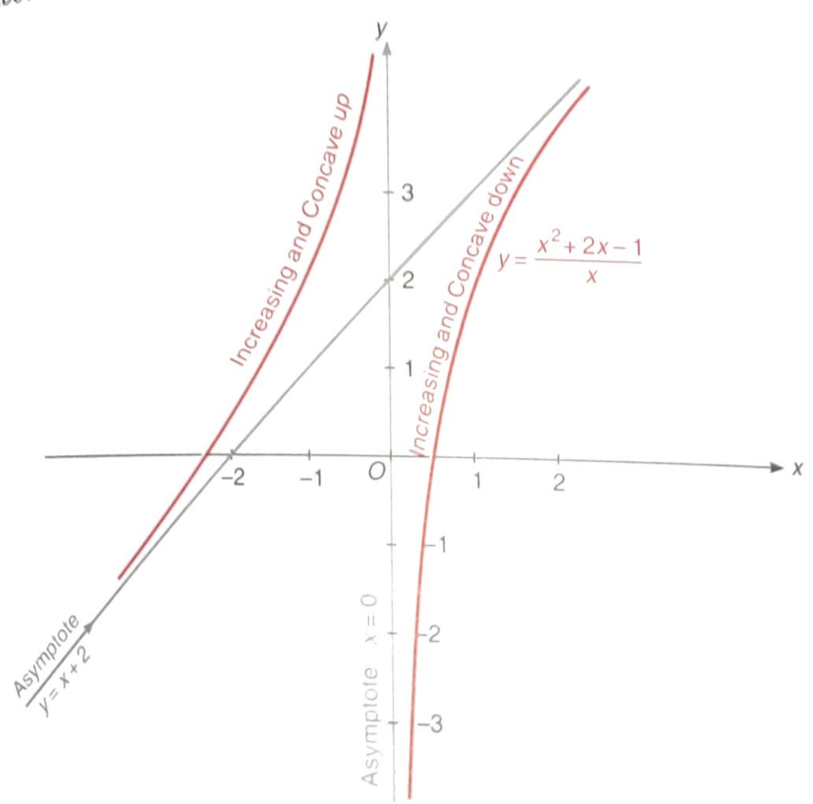


Fig. 3.6

3.1 (iv) Asymptote by inspection

If the equation of the curve be of the form $F_n + F_{n-2} = 0$, where F_n and F_{n-2} are expressions in x and y such that degree of $F_n = n$ and degree of $F_{n-2} \le n - 2$, then every linear factor equated to zero will give an asymptote if no two straight lines represented by any other factor of F_n is parallel or coincident with it.

EXAMPLE 1 *Find the asymptote of the curve $x^2y + xy^2 = a^3$.*

SOLUTION Here, the given curve is.

$$x^2y + xy^2 = a^3 \quad \text{or} \quad x^2y + xy^2 - a^3 = 0$$

This equation is of the form $F_n + F_{n-2} = 0$

Here, $F_3 = x^2y + xy^2$ and $F_0 = -a^3$

$\therefore$ By inspection the asymptotes are given by

$$x^2y + xy^2 = 0 \quad \text{or} \quad xy(x+y) = 0$$

$\therefore$ The asymptotes are $\mathbf{x = 0}$, $\mathbf{y = 0}$, $\mathbf{x + y = 0}$.

EXAMPLE 2 *Find the asymptote of the curve $y = x + \frac{1}{x}$ (by Inspection).*

SOLUTION Here, the given curve is $x^2 - xy + 1 = 0$

This equation is of the form $F_n + F_{n-2} = 0$

Here $F_2 = x^2 - xy$

$$F_0 = 1$$

$\therefore$ By inspection the asymptotes are given by

$$x^2 - xy = 0 \quad \text{or} \quad x(x-y) = 0$$

$\therefore$ The asymptotes are $\mathbf{x = 0}$ and $\mathbf{x - y = 0}$.

(v) Intersection of curve and its asymptote

An asymptote of curve of nth degree cut the curve in $(n-2)$ points provided the asymptote is not parallel to any asymptote.

Hence, if there be N asymptotes of the curve, then they cut the curve in $N(n-2)$ points.

Note The number of asymptotes of an algebraic curve of nth degree can not be more than n.

EXAMPLE 1 *Show the asymptote of the curve $xy(x^2 - y^2) + x^2 + y^2 - 1 = 0$ cut at 8 points.*

SOLUTION The equation of the curve is,

$$xy(x^2 - y^2) + x^2 + y^2 - 1 = 0 \qquad \text{...(i)}$$

Here $n = 4$

This equation is of the type $F_n + F_{n-2} = 0$

Hence, $F_n = xy(x^2 - y^2) = xy(x-y)(x+y)$

and $F_{n-2} = x^2 + y^2 - 1$

$\therefore$ $F_n = 0$

$\Rightarrow$ $x = 0$, $y = 0$, $x - y = 0$ and $x + y = 0$ are the equations of asymptotes.

The combined equation of the asymptotes is,

$$xy(x-y)(x+y) = 0 \qquad \text{...(ii)}$$

Subtracting Eq. (ii) from (i), we get

$$x^2 + y^2 - 1 = 0$$

Thus, intersection of curve and asymptotes lie on this curve since, there are 4 asymptotes, i.e., $N = 4$.

$\therefore$ **Point of intersection of curve and asymptotes = 4(4 − 2) = 8.**

3.1 (vi) Asymptote by expansion

If the equation of the curve is of the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \dots$$

Then $y = mx + c$ will be an asymptote of the given curve.

EXAMPLE 1 *Find the asymptote of the curve* $y^3 = x^2(x - a)$.

SOLUTION The curve is, $y^3 = x^2(x - a) = x^3\left(1 - \frac{a}{x}\right)$

$\Rightarrow$ $$y = x\left(1 - \frac{a}{x}\right)^{1/3} \quad \text{or} \quad y = x\left(1 - \frac{1}{3}\frac{a}{x} - \frac{1}{9}\frac{a^2}{x^2} \dots\right)$$

or $$y = x - \frac{a}{3} - \frac{1}{9}\frac{a^2}{x} \dots$$ which is of the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \dots$$

$\Rightarrow$ $$y = mx + c$$ is asymptote

Hence, $y = x - \frac{a}{3}$ **is asymptote of the given curve.**

EXAMPLE 2 *Find the asymptote for* $y = x + \frac{1}{x}$.

SOLUTION Here; $y = x + \frac{1}{x}$ is of the form,

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \dots$$

$\Rightarrow$ $y = x$ is asymptote of the curve

$$y = x + \frac{1}{x}.$$

Note Above method is useful to find oblique asymptote. Thus, students are adviced to find vertical and horizontal asymptote (*i.e.*, asymptote parallel to x-axis and y-axis).

3.1 (vii) The position of the curve with respect to an asymptote

Let the equation of the curve is of the form;

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \dots, \text{ then}$$

(a) The curve lies above the asymptote if

(i) $A \neq 0$ and, A and x have same signs

or (ii) $A = 0,\ B > 0$

or (iii) $A = 0,\ B = 0,\ C \neq 0$ and C and x have same signs and

(b) The curve lies below the asymptote if

(i) $A \neq 0$ and, A and x have opposite signs.

or (ii) $A = 0,\ B < 0$

or (iii) $A = 0,\ B = 0,\ C \neq 0$ and C and x have opposite signs.

EXAMPLE 1 *For the curve* $y^5 = x^5 + 2x^4$; *show;*

(i) *The curve lies above the asymptote* $y = x + \frac{2}{5}$, *if* $x < 0$.

(ii) *The curve lies below the asymptote* $y = x + \frac{2}{5}$, *if* $x > 0$.

SOLUTION The given curve is,

$$y^5 = x^5 + 2x^4$$

or

$$y^5 = x^5\left(1 + \frac{2}{x}\right)$$

$$y = x\left(1 + \frac{2}{x}\right)^{1/5}$$

$$y = x\left(1 + \frac{2}{5}\cdot\frac{1}{x} - \frac{8}{25}\cdot\frac{1}{x^2}\ldots\right) = x + \frac{2}{5} - \frac{8}{25x} + \ldots$$

∴ The asymptote is $y = x + \frac{2}{5}$;

(i) Now if $A = -\frac{8}{25}$ and x have same sign $\Rightarrow x < 0$. **Then the curve lie above the asymptote.**

(ii) Now if $A = -\frac{8}{25}$ and x have opposite sign $\Rightarrow x > 0$. **Then the curve lie below the asymptote.**

EXAMPLE 2 *For the curve* $y = x + \frac{1}{x}$ *show,*

(i) *The curve lies above the asymptote* $y = x$, *if* $x > 0$

(ii) *The curve lies below the asymptote* $y = x$, *if* $x < 0$

SOLUTION The given curve is, $y = x + \frac{1}{x}$, is of the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}\ldots$$

Thus, $y = x$ is the asymptote to $y = x + \frac{1}{x}$.

(i) Now if $A = 1$ and x have same sign $\Rightarrow x > 0$, **then the curve lies above the asymptote.**

(ii) Now if $A = 1$ and x have opposite sign $\Rightarrow x < 0$, **then the curve lies below the asymptote.**

3.2 SINGULAR POINTS

Here, we shall discuss the following

(i) Multiple points

(ii) Double points

Types of double points :

(a) Node (b) Cusp (c) Isolated point

(iii) Tangent at the origin.

(iv) Necessary conditions for existence of double points.

(v) Types of cusps.

3.2 (i) Multiple points

A point on a curve is said to be a multiple point of order r, if r branches of the curve pass through this point.

If P is the multiple point of order r, then there will be r tangents at P, one of each of the r branches. These r tangents may be real, imaginary, distinct, coincident.

3.2 (ii) Double points

A point on a curve is said to be a double point of the curve, if two branches of the curve pass through this point.

Double points have two tangents, they may be real, imaginary, distinct or coincident.

Types of Double points

(a) Node

If the two branches of a curve pass through the double point and the tangents to them at the point are real and distinct, then the double point is called **a node** as shown in Fig. 3.7.

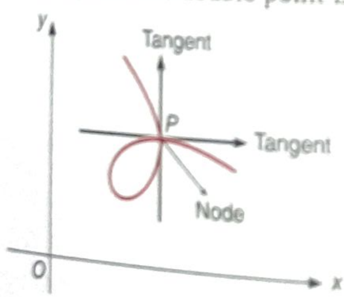


Fig. 3.7

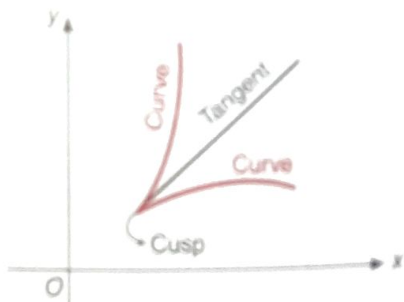


Fig. 3.8

(b) Cusp

If the two branches of the curve pass through the double point and the tangent to them are the point is real and coincident, then the double point is called **cusp** as shown in Fig. 3.8.

Cusps :

The graph of a continuous function $y = f(x)$ has a cusp at a point $x = c$ if the **concavity is same** on the both side of c and either.

1. $\lim\limits_{x \to c^-} f'(x) = \infty$ and $\lim\limits_{x \to c^+} f'(x) = -\infty$

OR

2. $\lim\limits_{x \to c^-} f'(x) = -\infty$ and $\lim\limits_{x \to c^+} f'(x) = \infty$ shown as:

1. $\lim\limits_{x \to c^-} f'(x) = \infty$ and $\lim\limits_{x \to c^+} f'(x) = -\infty$

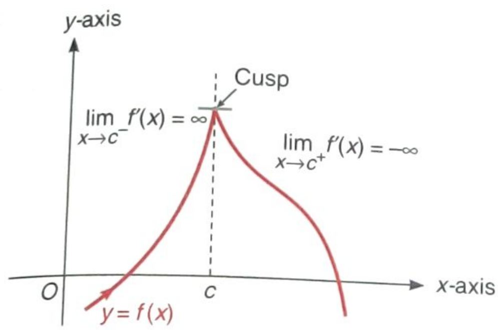


Fig. 3.9

2. $\lim\limits_{x \to c^-} f'(x) = -\infty$ and $\lim\limits_{x \to c^+} f'(x) = \infty$

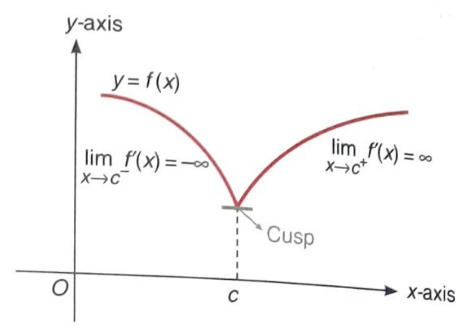


Fig. 3.10

Note A cusp can either be a local maximum (1) or a local minima as in (2).

(c) Isolated point

If there are no real point on the curve in the neighbourhood of a point P is called an **isolated or a conjugate point**.

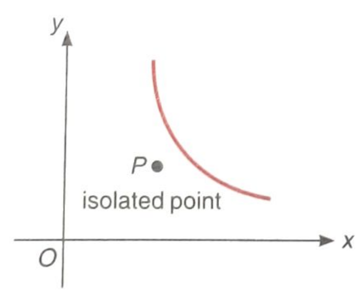


Fig. 3.11

3.2 (iii) Tangent at the origin

If an algebraic curve passes through the origin, the equation of tangent or tangents at the origin is obtained by equating to zero the lowest degree terms in the equation of the curve.

EXAMPLE 1 *Show that the curve* $y^2 = 4x^2 + 9x^4$ *has a node at origin and hence, sketch.*

SOLUTION The equation of the curve is,

$$y^2 = 4x^2 + 9x^4 \qquad \text{...(i)}$$

It passes through the origin.

Now, equating to zero the lowest degree terms of the given curve, *i.e.*,

$$y^2 - 4x^2 = 0$$

$$y = 2x \quad \text{and} \quad y = -2x$$

$\Rightarrow$ There are two real and distinct tangents $y = 2x$ and $y = -2x$. Thus, two branches of curve passes through origin (0, 0).

$\therefore$ origin is node. ...(ii)

Now to sketch;

(iii) Symmetric about x-axis, y-axis and origin.

(iv) As $x \to 0 \Rightarrow y \to 0$

(v) Domain $\in R$.

(vi) Range $\in R$.

Here, we shall discuss the behaviour of $y = x\sqrt{4 + 9x^2}$, $x \geq 0$ and use symmetry to construct $y^2 = x^2(4 + 9x^2)$.

(vii) $2y\dfrac{dy}{dx} = 8x + 36x^3 = 4x(2 + 9x^2)$

$\Rightarrow \quad \dfrac{dy}{dx} > 0$ for all $x, y > 0$

(viii) Also,

$$\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 4 + 54x^2$$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} > 0$ for all x

Thus, the graph for $y^2 = x^2(4 + 9x^2)$

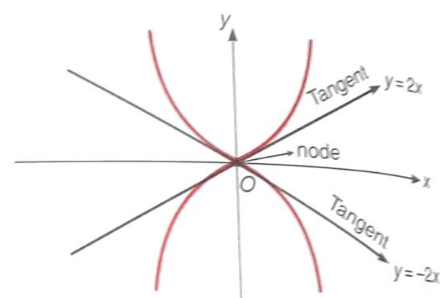


Fig. 3.12

EXAMPLE 2 *Show origin is a conjugate point for*

$$x^4 + y^3 + 2x^2 + 3y^2 = 0$$

SOLUTION The given curve is, $\quad x^4 + y^3 + 2x^2 + 3y^2 = 0$

It passes through origin.

$\therefore$ To find equation of tangent at origin equating the lowest degree term to zero.

i.e.,

$$2x^2 + 3y^2 = 0$$

$\Rightarrow$

$$y = \pm i\sqrt{\frac{2}{3}}\,x$$

which are imaginary tangents.

Hence, **origin is a conjugate point of the curve.**

3.2 (iv) Necessary conditions for the existence of double points

Let (x, y) be a point on the given curve $f(x, y) = 0$.

The necessary and sufficient conditions for (x, y) to be a double points are:

$$f = 0, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 \quad \text{at} \quad (x, y)$$

Now, if $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are not all zero, then,

(i) Double point will be a node if

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) > 0$$

or

$$f_{xy}^2 - f_{xx} f_{yy} > 0$$

(ii) The double point will be an isolated point, if

$$f_{xy}^2 - f_{xx} f_{yy} < 0$$

(iii) The double point will be a cusp if

$$f_{xy}^2 - f_{xx} f_{yy} = 0.$$

Here, if $f_{xx} = f_{xy} = f_{yy} = 0$ at (x, y), then it will be a multiple point of order greater than 2.

EXAMPLE 1 *For the curve* $x^3 + x^2 + y^2 - x - 4y + 3 = 0$, *find the double point and hence, whether the point is node or isolated point.*

SOLUTION Let $f(x, y) = x^3 + x^2 + y^2 - x - 4y + 3 = 0$

$\therefore$ $f_x = 3x^2 + 2x - 1$

$f_y = 2y - 4$ for a double point $f_x = 0$, $f_y = 0$

$\therefore$ $f_x = 0 \Rightarrow 3x^2 + 2x - 1 = 0$

or $x = \frac{1}{3}, -1$

$f_y = 0 \Rightarrow 2y - 4 = 0 \Rightarrow y = 2.$

$\therefore$ Possible double points are $\left(\frac{1}{3}, 2\right)$, $(-1, 2)$

$\because$ $f\left(\frac{1}{3}, 2\right) \neq 0$ and $f(-1, 2) = 0$

$\therefore$ **$f(-1, 2)$ is a double point.**

$$f_{xx} = 6x + 2 \Rightarrow f_{xx} \text{ at } (-1, 2) = -4$$

$$f_{xy} = 0 \Rightarrow f_{xy} \text{ at } (-1, 2) = 0$$

$$f_{yy} = 2 \Rightarrow f_{yy} \text{ at } (-1, 2) = 2$$

$\therefore$ $f_{xy} - f_{xx} f_{yy} = 0 - (-4)(2) = 8 > 0$

$\therefore$ **$(-1, 2)$ may be node.**

For shifting origin to $(-1, 2)$, substitute $x = X - 1$, $y = Y + 2$ in the given equation, we get,

$$X^3 - 2X^2 + Y^2 = 0$$

or

$$Y = \pm X\sqrt{2 - X}$$

For numerically small values of X, Y is real.

$\therefore$ **$(-1, 2)$ is a node on the given curve.**

EXAMPLE 2 *For the curve* $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$, *find the double point and hence, check whether node, cusp or isolated point.*

SOLUTION Let $f(x, y) = x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ ...(i)

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 4x + 2y + 5$$

$$f_y = \frac{\partial f}{\partial y} = 2x - 2y - 2$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x + 4$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\, \partial y} = 2$$

$$f_{yy} = \frac{\partial^2 f}{\partial y\, \partial x} = -2$$

For double points $f_x = f_y = f = 0$

$f_x = 0 \Rightarrow 3x^2 + 4x + 2y + 5 = 0$ ...(ii)

$f_y = 0 \Rightarrow 2x - 2y - 2 = 0$

or $2y = 2x - 2$ ...(iii)

Solving Eqs. (ii) and (iii), we get $3x^2 + 4x + 2x - 2 + 5 = 0$

$x = -1$

$\Rightarrow$ $x = -1,\ y = -2$ satisfies the given equation.

also

$\therefore$ **(– 1, – 2) is a double point.**

At (– 1, – 2), $f_{xx} = 6(-1) + 4 = -2$

$f_{xy} = 2,\ f_{yy} = -2$

$\therefore$ $f_{xy}^2 - f_{xx} f_{yy}$ at $(-1, -2) = 0$

$\therefore$ (–1, – 2) may be a cusp.

For shifting the origin to (– 1, – 2) substitute $x = X - 1,\ y = Y - 2$ in the given equation.

$$(X-1)^3 + 2(X-1)^2 + 2(X-1)(Y-2) - (Y-2)^2 + 5(X-1) - 2(Y-2) = 0$$

or $X^3 - X^2 + 2XY - Y^2 = 0$ ...(iv)

$\therefore$ $Y = X \pm X\sqrt{X}$

Y is real for all positive value of X.

$\therefore$ Two branches of (iv) pass through origin.

$\therefore$ Two branches of (i) pass through (– 1, – 2).

$\Rightarrow$ **(– 1, – 2) is a cusp.**

EXAMPLE 3 *Find for the curve* $y^2 = x \sin x$ *origin is node, cusp or isolated point.*

SOLUTION Let

$$f(x, y) = y^2 - x \sin x$$

$$f_x = -\sin x - x\cos x$$

$$f_y = 2y$$

$$f_{xx} = -\cos x + x \sin x - \cos x$$

$$f_{xy} = 0$$
$$f_{yy} = 2$$

$\therefore$ at $x=0$: $f_{xx} = -2,\ f_{xy} = 0,\ f_{yy} = 2$

$f_{xy}^2 - f_{xx} f_{yy}$ at $(0, 0)$

$$0 + 2(2) = 4 > 0$$

$$f_{xy}^2 - f_{xx} f_{yy} > 0$$

$\therefore$ **origin is node.**

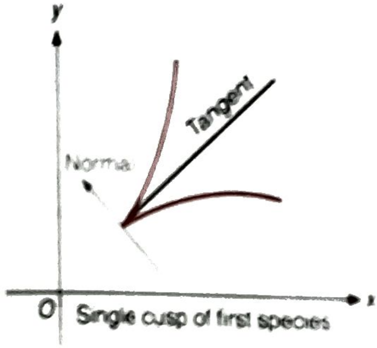
Types of cusps

When two branches of a curve pass through a cusp and the tangents at cusp are coincident. Therefore, normal to the branches at a cusp would also be coincident.

Cusp can be of five kinds

Single cusp

If the branches of the curve lie on the same side of the common normal, then the cusp is called a single cusp.



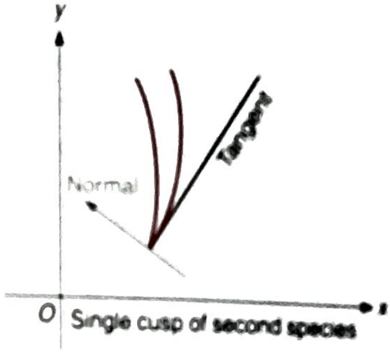


Fig. 3.13

Double cusp

If the branches of the curve lie on the both sides of the common normal, then the cusp is called double cusp.

Here, both the branches of the curve lie on the both sides of common tangent, then the cusp is of first kind.

Also if, the branches of the curve lie on the same side of the common tangent, then the cusp is called **cusp of second species or Ramphoid cusp.**

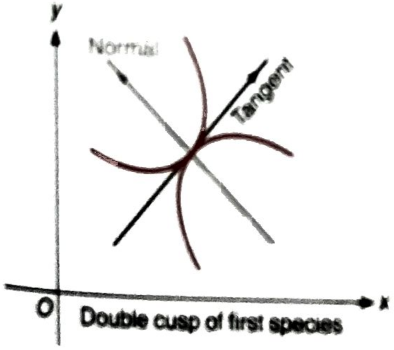


Fig. 3.14

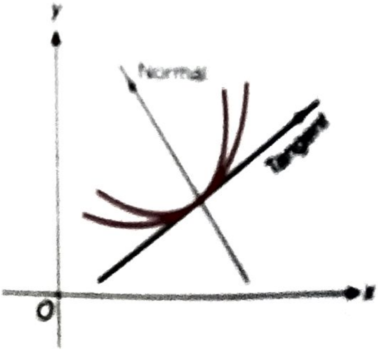


Fig. 3.15

(c) Point of oscu-inflexion

A double cusp of both the species is called a point of oscu-inflexion.

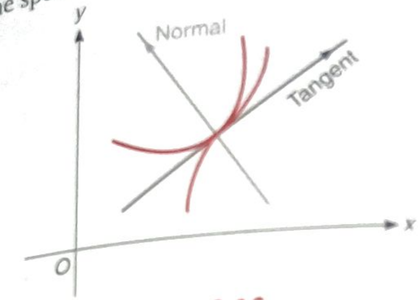


Fig. 3.16

EXAMPLE 1 *Sketch the curve* $y^2(a+x) = x^2(a-x)$.

SOLUTION Here, the curve is

$$y^2(a+x) = x^2(a-x) \qquad \text{...(i)}$$

1. The curve is **symmetrical about x-axis**.
2. Curve passes through origin and cuts the x-axis at a point $(a, 0)$.
3. Equating to zero the lowest degree terms of Eq. (i), we get the tangents at origin.

$$y^2 = x^2 \quad \text{or} \quad y = \pm x$$

$\therefore$

$\therefore$ **origin is node.**

4. Equating to zero the highest degree term, we get the asymptote

$$x + a = 0, \quad i.e., \quad x = -a.$$

5. From Eq. (i) $\qquad y = \pm x\sqrt{\dfrac{a-x}{a+x}}$

$\therefore$ y exists when $-a < x \le a \Rightarrow$ Domain $\in (-a, a]$.

6. As x increases from 0 to $a \Rightarrow y$ increases upto a point then decreases to zero.

$\Rightarrow$ y increases when $x \in [0, a]$

y decreases when $x \in (-a, 0]$

Thus, $y^2(a+x) = x^2(a-x)$ could be plotted as,

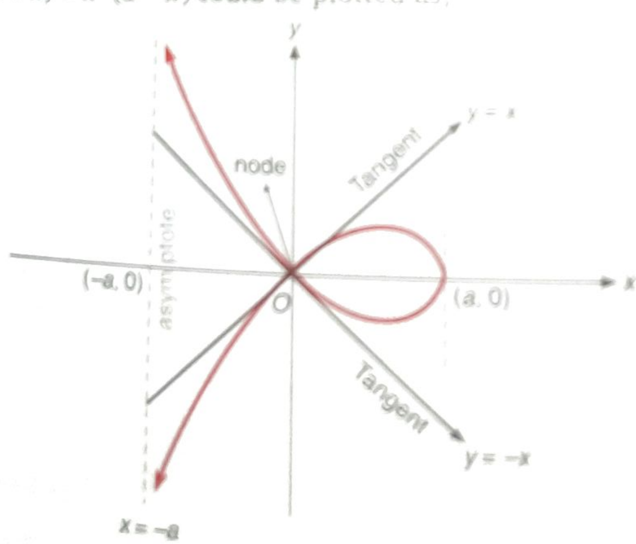


Fig. 3.17

3.3 REMEMBER FOR TRACING CARTESIAN EQUATION

1. Check symmetry

(a) A curve is symmetrical about x-axis, *i.e.*, y is replaced by $-y$ and curve remains same.

(b) A curve is symmetrical about y-axis, *i.e.*, $f(-x) = f(x)$.

(c) A curve is symmetrical about $y = x$, *i.e.*, on interchanging x and y curve remains same.

(d) A curve is symmetrical about $y = -x$. *i.e.*, on interchanging x by $-y$ and y by $-x$ curve remains same.

(e) The curve is symmetrical in opposite quadrants, *i.e.*, $f(-x) = -f(x)$.

2. Check for origin

Find whether origin lies on the curve or not.

If yes, check for multiple points (See Art. 3.2).

3. Point of intersection with x-axis and y-axis

Put $x = 0$ and find y, put $y = 0$ and find x. Also obtain the tangents at such points.

4. Asymptotes

Find the point at which asymptote meets the curve and equation of asymptote (see Art. 3.1)

5. Domain and range

To check in which part the curve lies.

6. Monotonicity and maxima minima

Find $\frac{dy}{dx}$ and check the interval in which y increases or decreases and the point at which it attains maximum or minimum.

7. Concavity and convexity

The interval in which,

$$\frac{d^2y}{dx^2} > 0$$

and

$$\frac{d^2y}{dx^2} < 0$$

Using all the above results we can sketch the curve

$$y = f(x).$$

SOME MORE SOLVED EXAMPLES

EXAMPLE 1 *Sketch the curve* $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

SOLUTION Here, the curve is $y^2 = \dfrac{x^2(a^2 - x^2)}{(a^2 + x^2)}$

1. The curve is symmetric about x-axis and y-axis {as on replacing y by $-y$ curve remains same and on replacing x by $-x$ curve remains same thus, symmetric about x and y-axis respectively).
2. It passes through origin and $y = \pm x$ are two tangents at origin. **Thus, the origin is node.**
3. It meets x-axis at $(a, 0)$, $(0, 0)$ and $(-a, 0)$ and meets y-axis at $(0, 0)$ only. The tangents at $(a, 0)$ and $(-a, 0)$ are $x = a$ and $x = -a$ respectively.
4. The curve has no asymptote.
5. Here,

$$y = \pm x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

∴ $$\text{Domain} \in [-a, a]$$

6. $$\frac{dy}{dx} = \frac{a^4 - 2a^2x^2 - x^4}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

$$\frac{dy}{dx} \to \infty \quad \text{as} \quad x \to \pm a$$

Also $$\frac{dy}{dx} = 0 \quad \text{when} \quad a^4 - 2a^2x^2 - x^4 = 0$$

i.e., $$\frac{dy}{dx} = \frac{a^4 - 2a^2x^2 - x^4}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

$$= \frac{-\{x^4 + 2a^2x^2 + a^4 - 2a^4\}}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

$$= \frac{-\{(x^2 + a^2)^2 - (\sqrt{2}a^2)^2\}}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

$$= \frac{-\{x - \sqrt{(-1 + \sqrt{2})}a\}\{x + \sqrt{(-1 + \sqrt{2})}a\}\{x^2 + (1 + \sqrt{2})a^2\}}{(a^2 + x^2)^{3/2}(a^2 - x^2)^{1/2}}$$

⇒ $$\frac{dy}{dx} = \begin{cases} 0; & x = \pm\sqrt{(-1 + \sqrt{2})}\, a \\ +\text{ ve}; & x \in (-\sqrt{(-1 + \sqrt{2})}\, a, \sqrt{(-1 + \sqrt{2})}\, a) \\ -\text{ ve}; & x \in (-a, -\sqrt{(-1 + \sqrt{2})}\, a) \text{ or } (\sqrt{(-1 + \sqrt{2})}\, a, a) \end{cases}$$

i.e., y increasing when $x \in (-\sqrt{(-1 + \sqrt{2})}\, a, \sqrt{(-1 + \sqrt{2})}\, a)$

and y decreases when $x \in (-a, -\sqrt{(-1 + \sqrt{2})}\, a)$ or $(\sqrt{(-1 + \sqrt{2})}\, a, a)$

OR

$$\frac{dy}{dx} > 0, \quad \text{when} \quad x \in (-(0.6)\,a, (0.6)\,a)$$

$$\frac{dy}{dx} < 0, \quad \text{when} \quad x \in (-a, -(0.6)\,a) \quad \text{or} \quad ((0.6)\,a, a)$$

where $\sqrt{-1+\sqrt{2}} = (0.6)_{\text{approx}}$

Thus, the curve for $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

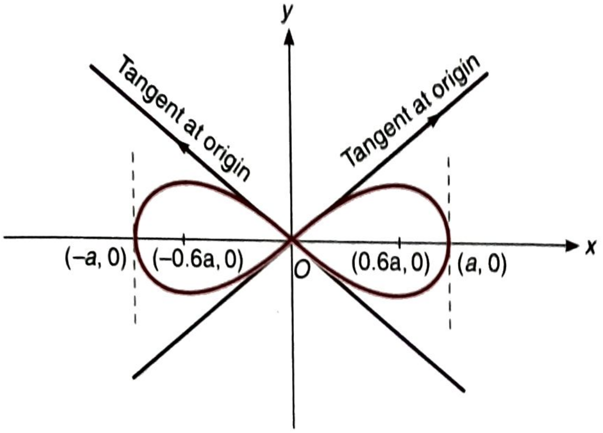


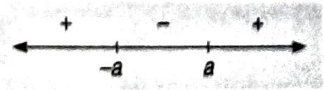
Fig. 3.18

EXAMPLE 2 *Sketch the curve* $y^2(x-a) = x^2(a+x)$.

SOLUTION Here, the curve is given by $y^2 = \dfrac{x^2(a+x)}{(x-a)}$

1. Symmetrical about x-axis only.
2. It passes through origin and $y^2 + x^2 = 0$, *i.e.*, $y = \pm ix$ are two imaginary tangents at origin. Thus, origin is **isolated point**.
3. It meets x-axis at $(-a, 0)$, $(0, 0)$ and y-axis at $(0, 0)$.
   The tangent at $(-a, 0)$ is $x = -a$.
4. $y = \pm(x-a)$ and $x = a$ are three asymptote.
5. $$y^2 = \frac{x^2(x+a)}{(x-a)} \Rightarrow y = \pm x\sqrt{\frac{x+a}{x-a}}$$

   Thus, for domain; $\dfrac{x+a}{x-a} \geq 0$ and $x \neq a$



   i.e., $x \leq -a$ and $x > a$

   or Domain $\in (-\infty, -a] \cup (a, \infty) \cup \{0\}$

6. $$\frac{dy}{dx} = \pm\frac{x^2 - ax - a^2}{(x-a)^{3/2}(x+a)^{1/2}} = \pm\frac{\left\{x - \frac{1}{2}(1+\sqrt{5})a\right\}\left\{x - \frac{1}{2}(1-\sqrt{5})a\right\}}{(x-a)^{3/2}(x+a)^{1/2}}$$

   $$\Rightarrow \quad \frac{dy}{dx} > 0, \quad \text{when} \quad x \in (-\infty, -a] \cup \left[\frac{1}{2}(1+\sqrt{5})a, \infty\right)$$

$$\frac{dy}{dx} < 0, \quad \text{when} \quad x \in \left(a, \frac{1}{2}(1+\sqrt{5})\, a\right]$$

Thus, the curve;

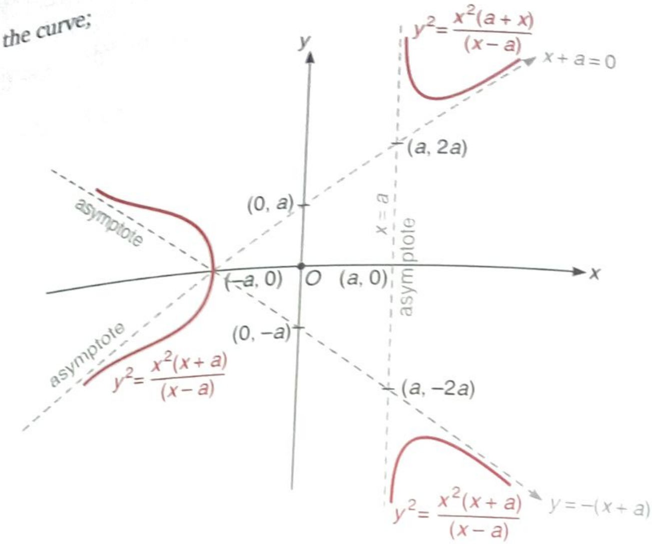


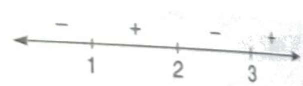
Fig. 3.19

EXAMPLE 3 *Sketch the curve* $y^2 = (x-1)(x-2)(x-3)$.

SOLUTION Here, $y^2 = (x-1)(x-2)(x-3)$

1. Symmetrical about x-axis.
2. It does not pass through origin.
3. It meets x-axis at $(1, 0)$ $(2, 0)$ and $(3, 0)$ but it does not meet y-axis.
4. No asymptote.
5. For domain: $(x-1)(x-2)(x-3) \geq 0$

$\Rightarrow$ **Domain** $\in$ **[1, 2]** $\cup$ **[3, ∞)**



6. $y = \pm\sqrt{(x-1)(x-2)(x-3)}$

$$\therefore \quad \frac{dy}{dx} = \pm\frac{(3x^2 - 12x + 11)}{2\sqrt{(x-1)(x-2)(x-3)}} = \pm\frac{3\left(x - \frac{6-\sqrt{3}}{3}\right)\left(x - \frac{6+\sqrt{3}}{3}\right)}{2\sqrt{(x-1)(x-2)(x-3)}}$$

$$= \pm\frac{3(x-1.42)(x-2.5)}{2\sqrt{(x-1)(x-2)(x-3)}} \quad \left\{\text{as } \frac{6-\sqrt{3}}{3} = 1.42 \text{ and } \frac{6+\sqrt{3}}{3} = 2.5/\text{approx}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} > 0, \quad \text{when} \quad x \in (1, 1.42) \cup (3, \infty)$$

$$\frac{dy}{dx} < 0, \quad \text{when} \quad x \in (1.42, 2)$$

Thus, the curve [illegible]

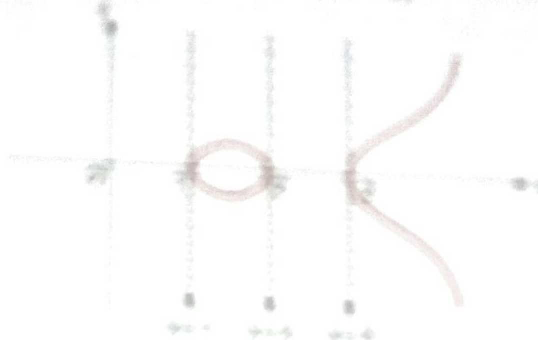


Fig. 5.20

EXAMPLE 4 Sketch the curve [illegible]

SOLUTION Here, [illegible]

(i) Symmetrical about both the axis.

(ii) It does not pass through origin.

(iii) x-intercepts are $(a, 0)$ and $(-a, 0)$.

The tangent at $(a, 0)$ is [illegible] and the tangent at $(-a, 0)$ is [illegible]

(iv) [illegible] are the two asymptotes.

(v) [illegible]

(i) [illegible]

(ii) [illegible]

Thus, the curve is [illegible]

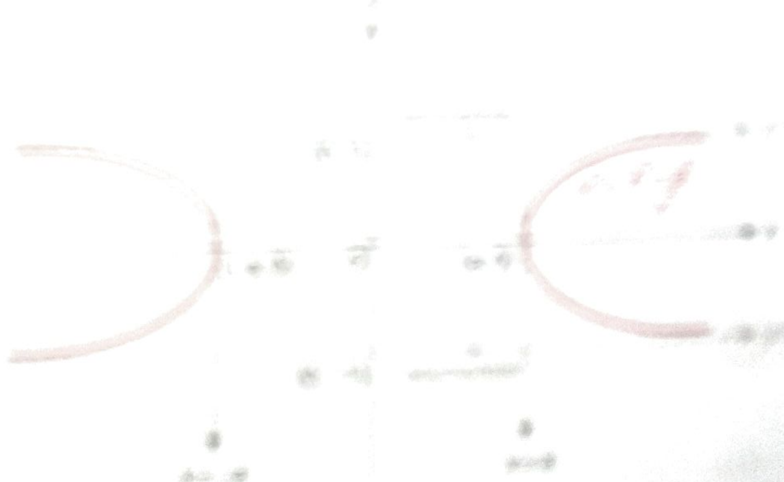


Fig. 5.21

Thus, the curve $y^2 = (x-1)(x-2)(x-3)$

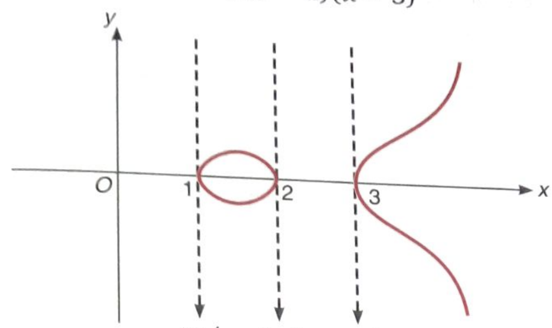


Fig. 3.20

EXAMPLE 4 *Sketch the curve* $y^2x^2 = x^2 - a^2$.

SOLUTION Here, $y^2 = \dfrac{x^2 - a^2}{x^2}$

1. Symmetrical about both the axis.
2. It does not pass through origin.
3. x-intercepts are $(a, 0)$ and $(-a, 0)$

   The tangent at $(a, 0)$ is $x = a$ and the tangent at $(-a, 0)$ is $x = -a$.
4. $y = \pm 1$ are the two asymptotes.
5. $$y = \pm \frac{\sqrt{x^2 - a^2}}{x} \quad \Rightarrow \quad \text{Domain} \in (-\infty, -a] \cup [a, \infty)$$

$$\frac{dy}{dx} = \pm \frac{a^2}{x^2\sqrt{x^2 - a^2}} \quad \Rightarrow \quad \frac{dy}{dx} > 0, \quad \text{when} \quad x \in (-\infty, -a) \cup (a, \infty)$$

Thus, the curve for $y^2 = \dfrac{x^2 - a^2}{x^2}$ is,

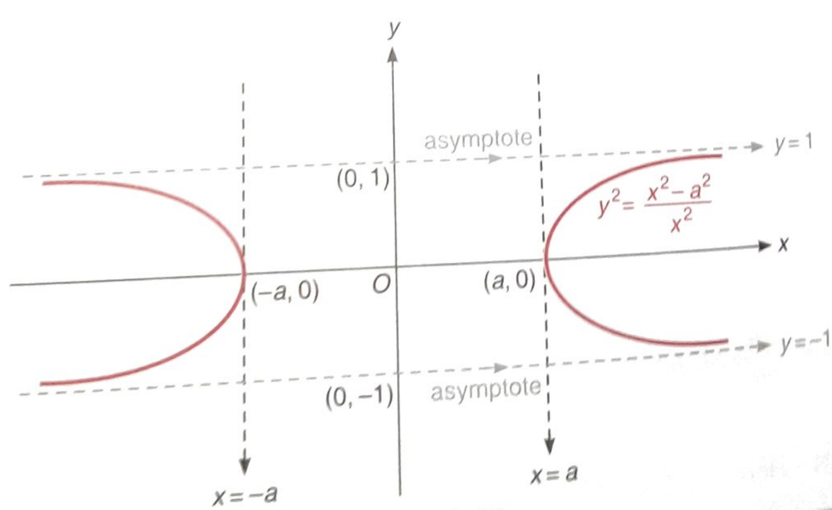


Fig. 3.21

EXAMPLE 5 *Sketch the curve* $y^2(x^2-1)=2x-1$.

SOLUTION Here, $y^2=\dfrac{2x-1}{x^2-1}$

1. Symmetrical about x-axis.
2. It does not pass through origin.
3. It meets x-axis in $\left(\dfrac{1}{2}, 0\right)$ and y-axis in $(0, 1)$ and $(0, -1)$ respectively.

   The tangent at $\left(\dfrac{1}{2}, 0\right)$ is $x=\dfrac{1}{2}$.
4. $x=1$, $x=-1$ and $y=0$ are three asymptotes.
5. $y^2=\dfrac{2x-1}{x^2-1}$ $\Rightarrow$ Domain $\in\left(-1, \dfrac{1}{2}\right]\cup(1,\infty)$
6. $y=\pm\sqrt{\dfrac{2x-1}{x^2-1}}$ $\Rightarrow$ $\dfrac{dy}{dx}=\pm\left(\dfrac{-x^2+x+1}{(2x-1)^{1/2}(x^2-1)^{3/2}}\right)$

$\Rightarrow$ $\dfrac{dy}{dx}<0$ when $x\in\left(-1, \dfrac{1}{2}\right]\cup(1,\infty)$

$\therefore$ **y is decreasing in its domain.**

Thus, the graph for $y^2=\dfrac{2x-1}{x^2-1}$ is,

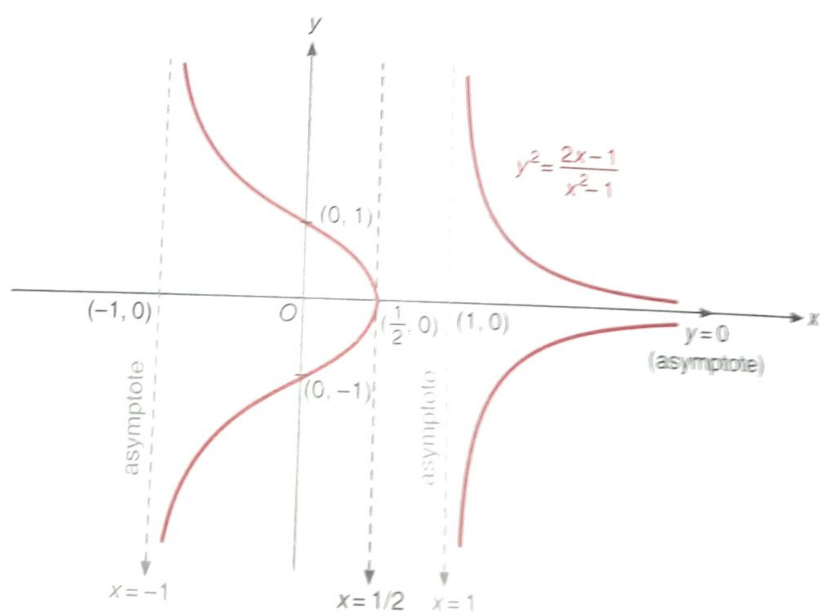


Fig. 3.22

EXAMPLE 6 *Sketch the curve :*

$$(x^2+y^2)x-a(x^2-y^2)=0;\ (a>0)$$

SOLUTION Here, $y^2=x^2\left(\dfrac{a-x}{a+x}\right)$

1. Symmetric about x-axis.
2. Origin lies on the curve and $y = \pm x$ are two tangents at origin. So, origin is node.
3. x-intercept are $(0, 0)$ and $(a, 0)$. The tangent at $(a, 0)$ is $x = a$.
4. $x = -a$ is the only asymptote.
5. $$y = \pm x\sqrt{\frac{a-x}{a+x}}$$

$\therefore$ Domain $\in (-a, a]$

6. $$\frac{dy}{dx} = \pm \frac{a^2 - ax - x^2}{(a+x)\sqrt{a^2 - x^2}}$$

$\Rightarrow \frac{dy}{dx} > 0$, when $x \in \left(-a, \frac{-1+\sqrt{5}}{2}a\right)$

$\Rightarrow \frac{dy}{dx} < 0$, when $x \in \left(\frac{-1+\sqrt{5}}{2}a, a\right)$.

Thus, the graph for

$y^2 = x^2\left(\frac{a-x}{a+x}\right)$ as shown in Fig. 3.23.

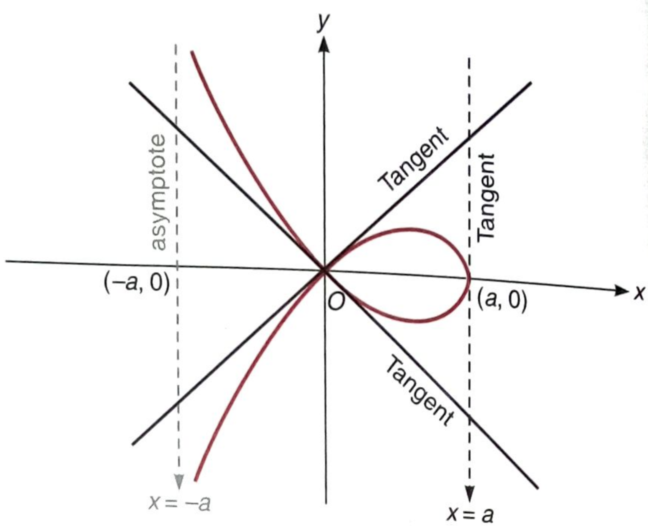


Fig. 3.23

EXAMPLE 7 *Sketch the curve* $x^3 + y^3 = 3ax^2 \quad (a > 0)$.

SOLUTION Here, $x^3 + y^3 = 3ax^2$

1. No line of symmetry.
2. Origin is cusp and $x = 0$ is tangent.
3. x-intercept, $(0, 0)$ $(3a, 0)$

   The tangent at $(3a, 0)$ is $x = 3a$.
4. $y = a - x$ is asymptote and the curve meets asymptote at $\left(\frac{a}{3}, \frac{2a}{3}\right)$.
5. Here;
$$x^3 + y^3 = 3ax^2$$
$$\Rightarrow \quad 3ax^2 > 0 \qquad \text{(as, } a > 0\text{)}$$
$$\therefore \quad x^3 + y^3 > 0$$

   i.e., x and y both cannot be negative (thus, curve would not lie in third quadrant).
6. $$y^2 \frac{dy}{dx} = x(2a - x)$$

$\Rightarrow \frac{dy}{dx} > 0$, when $x \in (0, 2a)$

$\frac{dy}{dx} < 0$, when $x \in (-\infty, 0) \cup (2a, \infty)$

Thus, the curve $y^3 + x^3 = 3ax^2$ is,

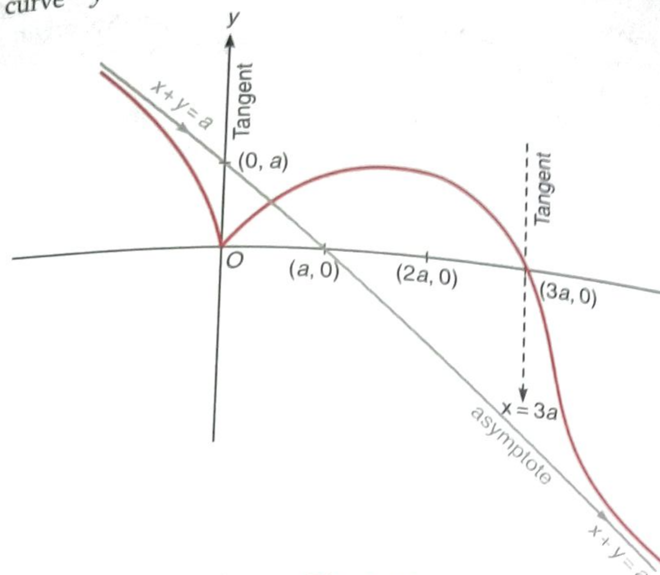


Fig. 3.24

EXAMPLE 8 *Sketch the curve with parametric equation* θ.

$$x = a(\theta + \sin\theta), \quad y = a(1 + \cos\theta); \quad x \in (-\pi, \pi).$$

SOLUTION Here, $x = a(\theta + \sin\theta)$ and $y = a(1 + \cos\theta)$ gives the following table for x and y with θ.

θ	$-\pi$	0	π
x	$-a\pi$	0	$a\pi$
y	0	$2a$	0

So, that we have,

$$-\pi \le \theta \le 0$$

$\Rightarrow$ (x, y) starting from $(-a\pi, 0)$ moves to the right and upwards to $(0, 2a)$.

$$0 \le \theta \le \pi$$

$\Rightarrow$ the point (x, y) starting from $(0, 2a)$ moves to the right and downward to $(a\pi, 0)$.

Also
$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

and
$$\frac{dy}{d\theta} = -a\sin\theta$$

Now, $\dfrac{dx}{d\theta} = 0$ if $\theta = \pi, -\pi$

$$\frac{dy}{dx} = -\frac{\tan\theta}{2},$$

except for the values $\mp\pi$ of θ for which $\dfrac{dx}{d\theta}=0$.

Also, tangent at $\theta=\pi$ and $\theta=-\pi$ are $x=\pi$ and $x=-\pi$.

Thus, the curve for $x=a(\theta+\sin\theta)$ and $y=a(1+\cos\theta)$.

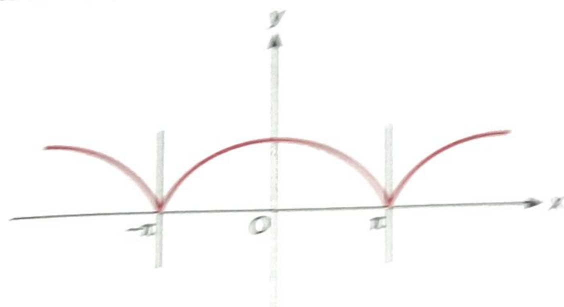


Fig. 3.25

EXAMPLE 9 *Sketch the curve :* $x^5+y^5=5a^2xy^2$.

SOLUTION Here;

1. The curve is symmetrical in opposite quadrants.
2. The curve passes through origin and $x=0$, $y=0$ are tangents. Thus, origin is node.
3. It meets coordinate axis at origin.
4. $x+y=0$ is an asymptote.
5. On transfering to polar coordinates, we get

$$r^2=\frac{5a^2\cos\theta\sin\theta}{\cos^5\theta-\sin^5\theta}$$

when, $\theta=0$, $r=0$ when, $\theta=\dfrac{\pi}{2}$, $r=0$

As θ increases from $\dfrac{\pi}{2}$ to $\dfrac{3\pi}{4}$, r^2 is negative and hence r is imaginary.

$\therefore$ no portion of the curve lies in this region.

At $\theta=\dfrac{3\pi}{4}$, $r=\infty$ as θ increases from $\dfrac{3\pi}{4}$ to π $\Rightarrow$ r decreases from ∞ to 0.

$\therefore$ Curve $x^5+y^5=5a^2xy^2$

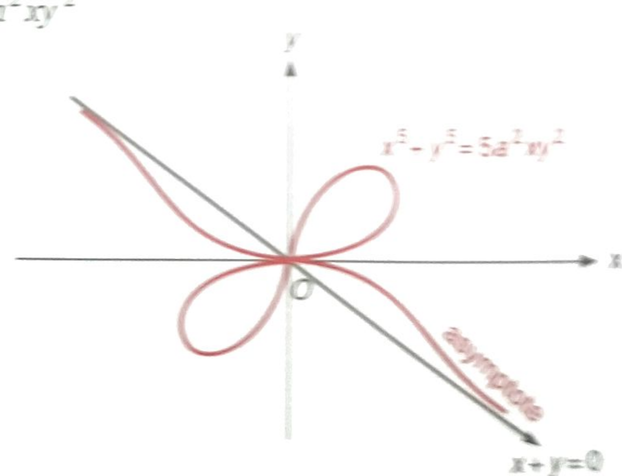


Fig. 3.26

EXAMPLE 10 *Sketch the curve* $y^4 - x^4 + xy = 0$.

SOLUTION Here, $y^4 - x^4 + xy = 0$

1. No line of symmetry.
2. It passes through origin two tangents at (0, 0) as $x = 0$ and $y = 0$,

   $\therefore$ **origin is node.**
3. It cuts the coordinate axes at the origin only.
4. $y = x$, $y = -x$ are its asymptotes.
5. Converting into polar coordinates,

$$r^2 = \frac{1}{2}\tan 2\theta$$

6. When, $0 < \theta < \frac{\pi}{4}$ or $0 < 2\theta < \frac{\pi}{2}$ $\Rightarrow$ r^2 increases from 0 to ∞.

   When, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < 2\theta < \pi$ $\Rightarrow$ r^2 is negative,

   $\therefore$ no curve when $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

   Hence, the curve

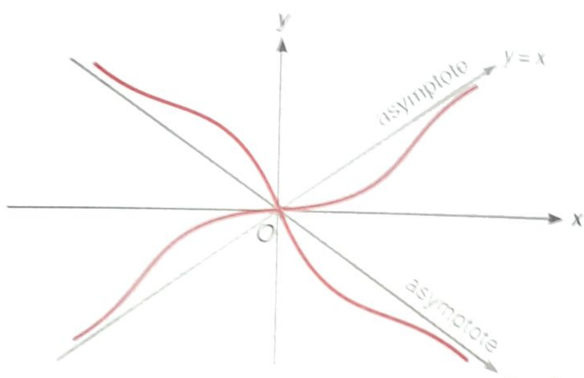


Fig. 3.27

EXERCISE

Plot the Curves :

1. $y = 1 + x^2 - \frac{1}{2}x^4$.
2. $y = (x+1)(x-2)^2$
3. $y = \frac{2}{5}x - \frac{1}{2}x^3 + \frac{1}{10}x^5$
4. $y = (1 - x^2)^{-1}$
5. $y = \frac{x^4}{(1+x)^3}$
6. $y = \frac{(1+x)^4}{(1-x)^4}$
7. $y = \frac{x^2(x-1)}{(x+1)^2}$
8. $y = \frac{x}{(1-x^2)^2}$
9. $y = 2x - 1 + \frac{1}{(x+1)}$
10. $y = \frac{x^2+1}{x^2-1}$
11. $y = \frac{a^2x}{a^2+x^2}$
12. $y^2 = x^2\left(\frac{a+x}{b-x}\right)$
13. $y = \frac{8a^3}{x^2+4a^2}$
14. $y = \frac{\cos x}{\cos 2x}$